보험회사 내부모형 개발 및 적용방안

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Outline

요구자본

→ A one-year risk horizon – 현재: $\tau = 0$, 미래: $\tau = 1$

→ Ignoring all the complications, **Available Capital**:

$$C_\tau = [\text{MV Assets}]_\tau - [\text{MV Liabilities}]_\tau, \ \tau \in \{0, 1\}$$

Solvency – intuition

An insurer is solvent if

$$\mathbb{P}\left(C_1 < 0\right) \leq .5\%$$

So...

- 1년뒤($\tau = 1$) 보험회사가 지급능력이 충분하기 위해서 현재 적립해야할 자본은?
- 현재 보험회사의 보유 자본: C_0 (i one-year risk-free rate @ $\tau = 0$):

$$\mathbb{P}(-C_1 > 0) = \mathbb{P}\left(\frac{-C_1}{1+i} > 0\right) = \mathbb{P}\left(C_0 - \frac{C_1}{1+i} > C_0\right) = ?$$

- C_0는 충분한가?
- 장래 손실을 충분하게 흡수하기 위하여 보험회사는 아래의 조건을 만족하는 x만큼 자본을 보유하고 있어야 함:

$$\mathbb{P}\left(C_0 - \frac{C_1}{1+i} > x\right) \leq .5\%$$

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Definition of SCR

→ We need to determine the 99.5%-quantile of $-C_1$, where

$$C_1 = [\text{MV Assets}]_1 - \mathbb{E}^{\mathbb{Q}}\left[\text{Disc. Fut. Policyholder CFs} | \mathcal{F}_1\right]$$

is an $\mathcal{F}_1$-measurable random variable.

→ 즉 C_1 의 99.5% quantile을 찾기 위해서는 그 분포 (distribution)가 필요함.

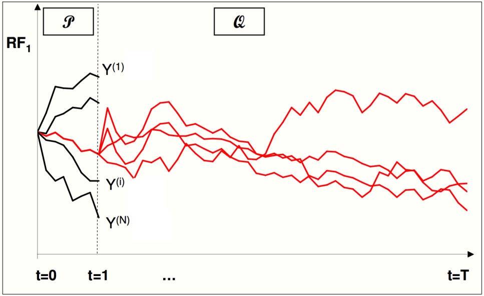
→ 그 분포를 찾기 위해서는 기댓값의 계산이 필요함.

→ It is Mostly unfeasible to find closed-form expressions for the required expectations.

→ 수치적 방법, Monte Carlo method

Methods

- **Estimating the SCR**: 리스크 호라이즌에서 발생한 각 시나리오 별로 다시 다량의 시나리오를 발생시켜야 함 (reevaluation of the company's assets and liabilities)
  → Nested Computation Structure



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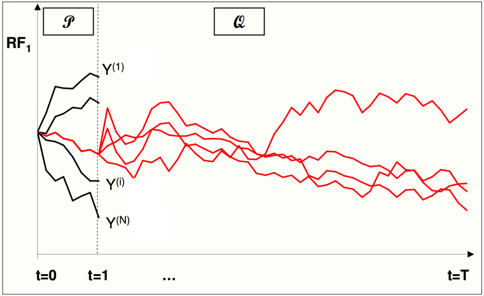

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Model Framework

- d-dimensional Markov process $Y = (Y_t)_{t \in [0,T]}$ drives assets and liabilities
- $\tilde{\mathbb{Q}}$ risk-neutral measure with respect to Numéraire N_t (e.g., 무이표 채권 또는 무위험 bank account)
- Single cash-flow: $X_T = X_T(Y_T)$
- $C_\tau = [\mathrm{MVA}]_\tau - N_\tau \mathbb{E}^{\tilde{\mathbb{Q}}}[X_T | Y_s, s \in [0,\tau]]$ ($N_T = 1$ for convenience)
- **Typically:**

$$[\mathrm{MVL}]_\tau = N_\tau \times \underbrace{\mathbb{E}^{\tilde{\mathbb{Q}}}[X_T | Y_\tau]}_{\text{To be approximated}}$$

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Methods

- 즉, $\mathbb{E}^{\tilde{\mathbb{Q}}}[X_T | Y_\tau]$ 각 outer 시나리오 별로 추정되어야 함.
- 두 가지 방법:
  1. $\mathbb{E}^{\tilde{\mathbb{Q}}}[X_T | Y_\tau] = g(Y_\tau)$를 직접 근사
  2. 미래 현금 흐름 자체$(X_T(Y_T))$를 근사, 이후 그 근사값의 기댓값을 계산? (in a closed-form hopefully)

LSM structure and information

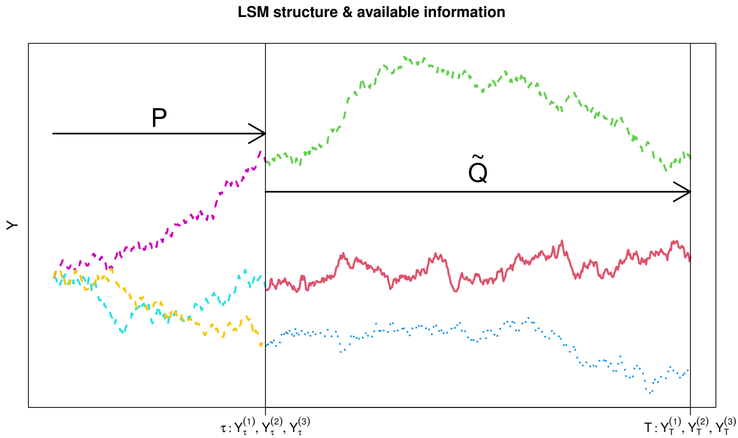

Figure 1: P: Real measure, $\tilde{Q}$: Pricing measure

Approximation Strategies

$g(Y_\tau)$ and $X_T(Y_T)$를 근사하는 방법은 적어도 두 가지가 존재함:

- LSM now: $\mathbb{E}^{\tilde{\mathbb{Q}}}\left[X_T | Y_\tau\right] = g(Y_\tau) \approx \sum_{j=1}^{M} \beta_j h_j(Y_\tau)$,
- LSM later: $X(Y_T) \approx \sum_{k=1}^{M} \alpha_k \varphi_k(Y_T)$
  $\rightarrow g(Y_\tau) \approx \sum_{k=1}^{M} \alpha_k \mathbb{E}^{\tilde{\mathbb{Q}}}[\varphi_k(Y_T)|Y_\tau]$

where $\{h_j\}_{j=1}^{M}$ and $\{\varphi_k\}_{k=1}^{M}$ are the sets of basis functions, and $\{\beta_j\}_{j=1}^{M}$ and $\{\alpha_k\}_{k=1}^{M}$ are the sets of corresponding coefficients to be *ESTIMATED* via the least-squares regression (다중선형 회귀분석).

LSM Now and LSM Later

- $X_T(Y_T)$를 $span\{h_1(Y_\tau), ..., h_M(Y_\tau)\}$에 직교 사영하면 (선형회귀분석을 통하여)

$$g(Y_\tau) \approx \sum_{j=1}^{M} \hat{\beta}_j h_j(Y_\tau)$$

LSM Now estimator.

- $X_T(Y_T)$를 $span\{\varphi_1(Y_T), ..., \varphi_M(Y_T)\}$에 직교 사영하면 (선형회귀분석을 통하여)

$$g(Y_\tau) \approx \sum_{k=1}^{M} \hat{\alpha}_k \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\varphi_k(Y_T) | Y_\tau\right]$$

LSM Later estimator.

Critical requirement for LSM later

- LSM later를 실행하기 위해서는 각 basis 함수의 기댓값이 알려져 있어야 함

$$g(Y_\tau) \approx \sum_{k=1}^{M} \hat{\alpha}_k \underbrace{\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\varphi_k(Y_T) | Y_\tau\right]}_{\text{How to compute?}} = \sum_{k=1}^{M} \hat{\alpha}_k \phi_k(Y_\tau).$$

where $\phi_k(Y_\tau)$ is the *closed-form expression* for $\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\varphi_k(Y_T) | Y_\tau\right]$.

1. This condition is the culprit in making regression later impractical.
2. A martingale condition: $\mathbb{E}^{\tilde{\mathbb{Q}}}\left[\varphi_k(Y_T) | Y_\tau\right] = \varphi_k(Y_\tau)$??

Formulation – Operator setting

- **Key change in perspective**: Define operator for $\mathbb{E}^{\tilde{\mathbb{Q}}}[\cdot|Y_\tau]$:

$$\mathcal{V} : \mathcal{L}^2\left(\mathbb{R}^d, \mathcal{B}, \tilde{\mathbb{Q}}\right) \to \mathcal{L}^2\left(\mathbb{R}^d, \mathcal{B}, \mathbb{P}\right)$$

Two LSMs are related to orthogonal projections:

- Regression now: $\mathcal{V}_F^N = P^N \mathcal{V} = \sum_{h=1}^M \langle \mathcal{V} X_T, h_j(Y_\tau) \rangle_{\mathcal{L}^2(\mathbb{P})} h_j$
- Regression later: $\mathcal{V}_F^L = \mathcal{V} P^L = \sum_{k=1}^M \langle X_T, \varphi_j(Y_T) \rangle_{\mathcal{L}^2(\tilde{\mathbb{Q}})} \mathcal{V} \varphi_k$

How to choose a good basis?

Choosing GOOD basis functions amounts to finding the basis functions that minimize the distance between $\mathcal{V}$ and the finite rank operator.

Robust Basis Functions

Definition (Regression now)

We call the set of basis functions $\{e_1^*, e_2^*, ..., e_M^*\}$ *robust* for the regression now in $\mathcal{L}^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P})$ if

$$\{h_1^*, h_2^*, ..., h_M^*\} = \text{arginf}_{\{h_1, h_2, ... h_M\}} \sup_{||X_T||=1} ||\mathcal{V}X_T - \mathcal{V}_F^N X_T||$$

Definition (Regression later)

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Compact valuation operator

Assume that there exists a transition density $\pi_{Y_T|Y_\tau}(y|x)$ for Y_T given Y_τ. Moreover,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \pi_{Y_T|Y_\tau}(y|x)\,\pi_{Y_\tau|Y_T}(x|y)\,dx\,dy < \infty,$$

where $\pi_{Y_\tau|Y_T}(x|y)$ is the reverse transition density.

Lemma

The operator $\mathcal{V}$ is an HS operator and, therefore, compact.

Singular Value Decomposition

If $\mathcal{V}$ is *compact*, the operator has a *singular value decomposition* $(\omega_i, s_i, \psi_i)_{i\geq 1}$:

$$\mathbb{E}^{\tilde{\mathbb{Q}}}[f(Y_T)|Y_\tau] = \sum_{i=1}^{\infty} \omega_i < f, s_i >_{\mathcal{L}^2(\tilde{\mathbb{Q}})} \psi_i(Y_\tau).$$

where $f \in \mathcal{L}^2(\mathbb{R}^d, \mathcal{B}, \tilde{\mathbb{Q}})$, $\omega_1 \geq \omega_2 \geq \cdots$ are singular values, and $\{s_i\}$ and $\{\psi_j\}$ are the right and left singular functions, respectively. In particular,

- $\{s_i\}$ forms a complete orthonormal basis for $\mathcal{L}^2(\mathbb{R}^d, \mathcal{B}, \tilde{\mathbb{Q}})$.
- $\{\psi_i\}$ forms a complete orthonormal basis for $\mathcal{L}^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P})$.
- $\mathbb{E}^{\tilde{\mathbb{Q}}}[s_j(Y_T)|Y_\tau] = \omega_j \psi_j(Y_\tau)$ for $j \geq 1$.

Robustness of Singular Functions

Proposition

Assume the operator $\mathcal{V}$ is compact. Then for each M,

- *the left singular functions* $\{\psi_1, \psi_2, \ldots, \psi_M\} \in \mathcal{L}^2(\mathbb{R}^d, \mathcal{B}, \mathbb{P})$ *are robust basis functions. For a fixed cash flow model, we obtain* $\beta_j = \omega_j \langle X_T, s_j \rangle$.
- *the right singular functions* $\{s_1, s_2, \ldots, s_M\} \in \mathcal{L}^2(\mathbb{R}^d, \mathcal{B}, \tilde{\mathbb{Q}})$ *are robust basis functions. For a fixed cash flow model, we obtain* $\alpha_k = \langle X_T, s_k \rangle$.

Proposition (Joint Convergence of LSMs)

The MSEs of two LSM algoritms are given by

$$O_{now}\left(\frac{M}{N} + \omega_{M+1}^2\right) \quad \text{and} \quad O_{later}\left(\frac{M}{N}\omega_M^2 + \omega_{M+1}^2\right),$$

respectively.

SVD for Gaussian Transition Densities

- Assume
$$\begin{pmatrix} Y_\tau \\ Y_T \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_\tau \\ \mu_T \end{pmatrix}, \quad \begin{pmatrix} \Sigma_\tau & \Gamma \\ \Gamma' & \Sigma_T \end{pmatrix}\right].$$
- Then, $\mathcal{V}$ is compact.
- Denote by $P\Lambda P'$ and $Q\Lambda Q'$ the eigenvalue decompositions of $\Sigma_\tau^{-1/2} A \Sigma_\tau^{1/2}$ and $\Sigma_T^{-1/2} B \Sigma_T^{1/2}$ with $P'P = Q'Q = I$ where $A = \Gamma\Sigma_T^{-1}\Gamma'\Sigma_\tau^{-1}$, and $B = \Gamma'\Sigma_\tau^{-1}\Gamma\Sigma_T^{-1}$.
- Eigenvalues ($diag(\Lambda)$): $\lambda_1 \geq \cdots \geq \lambda_d$.
- $z^P(y) = P'\Sigma_\tau^{-1/2}(y - \mu_\tau)$ and $z^Q(x) = Q'\Sigma_T^{-1/2}(x - \mu_T)$.

Proposition

The singular values of $\mathcal{V}$ are given by $\omega_m = \Pi_{i=1}^d \lambda_i^{k_i/2}$ where $m = (k_1, ..., k_d) \in \mathbb{N}_0^d$, and the corresponding right and left singular functions are:

$$s_m(x) = \Pi_{i=1}^d h_{k_i}(z_i^Q(x)) \text{ and } \psi_m(y) = \Pi_{i=1}^d h_{k_i}(z_i^P(y)).$$

GMIB 특약을 보유한 변액연금

VA를 보유한 x세인 피보험자가 $x+T$세에 생존하면,

$$\max\left\{S_T,\ \frac{G_T}{a^*_{x+T}(0)}\,a_{x+T}(T)\right\},$$

where

$$dS_t = S_t\left(m\,dt + \sigma_S\,dW^S_t\right),$$
$$dr_t = \alpha(\gamma - r_t)\,dt + \sigma_r\,dW^r_t,$$
$$d\mu_{x+t} = \kappa\mu_{x+t}\,dt + \delta\,dW^\mu_t.$$

t시점에서의 VA 시장가격(market-consistent value)은

$$V(\tau) = \underbrace{{}_{T-\tau}E_{x+\tau}\ \mathbb{E}^{\tilde{\mathbb{Q}}}\left[\overbrace{\max\left\{S_T,\ G_T/a^*_{x+T}(0)\times a_{x+T}(T)\right\}}^{\text{via LSM later}}\middle|\,Y_\tau\right]}_{\text{via LSM now}}.$$

Single Roll-Up GMIB

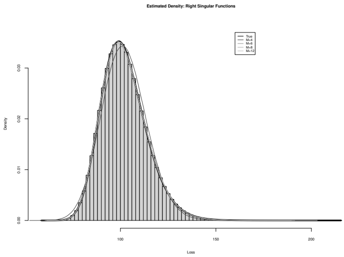
만약

$$G_T = S_0 \times (1 + m_g)^T,$$

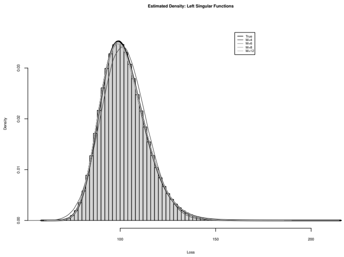
$d = 3$ 이고 모든 변량들은 정규분포를 따르게 됨.

- $\tau = 1$,
  $S_0 = 100, m = 5\%, \sigma_S = 18\%, r_0 = 2.5\%, \alpha = 25\%, \gamma = 2\%, \sigma_r = 1\%, \lambda = 2\%, x = 55, \mu_{55} = 1\%, \kappa = 7\%, \delta = 0.12\%.$
  $\rho_{12} = -30\%, \rho_{13} = 6\%, \rho_{23} = -4\%.$
- 다항함수를 기저함수로 두고, $M = 45, N = 45 \times 10^6$ 를 이용하여 C_τ 의 quasi-exact distribution 을 얻음.

Single Roll-Up GMIB – Dists.



LSM later



LSM now

Figure 2: Estimated densities via two LSM algorithms

Single Roll-Up GMIB – Performances

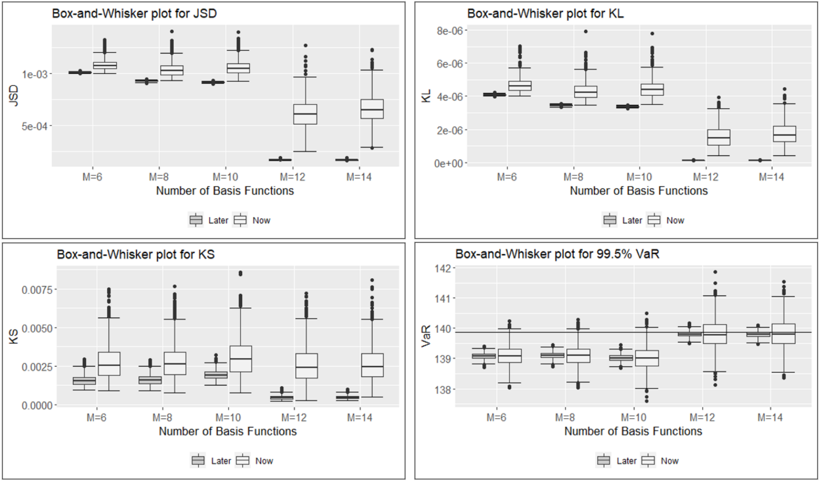


Figure 3: Statistical Divergences and Risk Measure of two LSMs, $N = 3,000,000$

Single Roll-Up GMIB – MSEs

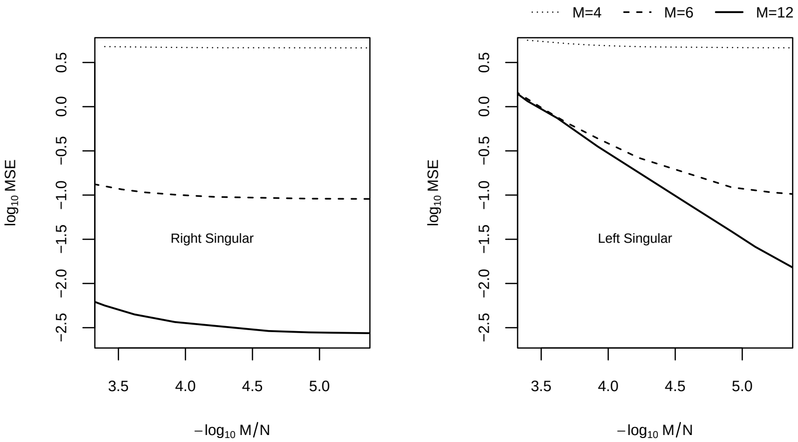


Figure 4: MSEs of two LSMs

Conclusion

- 위험 측정 및 요구 자본량 계산을 위한 LSM 방법은 실무에서 널리 사용되고 있음
  1. 낮은 계산량을 요구
  2. 상대적으로 이해하기 쉬움 (compared to neural networks)
- Questions:
  1. 어떤 기저함수를 몇개를 사용해야 하는가?
     $\rightarrow$ SVD of valuation operator
  2. LSM later or now?
     $\rightarrow$ LSM later는 LSM now에 비해 성능이 우수하지만, 이를 적용하기 위해서는 기저 함수의 기댓값이 알려져 있어야 함. 일반적으로는 강건성(robustness) 측면에서 LSM now가 LSM later보다 더 널리 사용됨.
- Extensions: Non-linear/ML models, risk-measure specific approaches such as importance sampling technique

Thank You

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