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A Method of Hedging Mortality Rate Risks in Endowment Product Development*

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Forecasting mortality rate changes in the future is important and necessary for insurance businesses. An interesting observation is that mortality rates for a few age groups have improved recently and that other mortality rate risks may exist. If the life table constructed from a mortality model, which predicts mortality rates lower than those actually experienced by the life insurance policy holders, then the company will face losses from the sales of life insurance contracts.

As a hedging strategy, the insurance company may promote the sale of polices, such as annuities or pure endowments, to offset the losses from the life insurance sales. We present a method of hedging mortality rate risks for the development of endowment policies using the hedge ratios of pure endowment in order to offset the losses from term life insurance. We also demonstrate a hedging strategy using the stochastic force of a mortality model, which is resulted from Malliavin calculus.

Key words: Mortality Rate Risks, Force of Mortality, Hedge Ratios, Stochastic Mortality Rate Models, Malliavin Calculus

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I. Introduction

Forecasting mortality rate changes in the future is necessary because mortality rate improvements pose a challenge for pricing and reserving in life insurance and for the management of public pension regimes. An interesting observation is that mortality rates for a few age groups have improved recently. Watts et al.(2006) showed a statistical analysis of advanced age mortality data, using extreme value models to quantify the upper tail of the distribution of human life spans. We may also refer to a few papers, such as Friedland(1998), Gutterman and Vanderhoof(1998), Tuljapurkar(1998), Rogers(2002), Kang et al.(2006), and Arnold and Sherris(2013), which discussed the trends in mortality changes and forecasting. Recently, Wade(2010), and Montambeault and Menard(2010) published papers regarding mortality projections for social security programs in the United States and Canada. They showed tremendous reductions in mortality at all ages for both males and females.

The same mortality trend can be observed in South Korea. Based on the “Abridged life table” from the Statistics of Korea(from 1992 to 2011) and the “Calculation results and methods for the referred mortality rate in life insurance FY2011” from the Korea Insurance Development Institute, the future mortality rate can be estimated using a modified exponential growth model. In order for the estimated 2060 mortality rate by age cohort to converge on the future life table provided by the Statistics of Korea, a lower limit is imposed in a cohort component method. Table 1, which can be found in Seo(2013), summarizes the accumulated improvement rate for different age groups of males from the modified exponential growth model. The accumulated percentage for the improvement rate of each age group increases with time, but does not evenly increase by age cohort.

Based on Table 1, we can generally observe that the accumulated improvement rates for the estimated male mortality of older age groups are remarkably enhanced. The age group from 75 to 79 years showed a particularly large improvement. This finding implies that the expected life expectancy is increasing. However, improvement rates for the 45 to 54 age groups performed better than those for people from 55 to 60. It is not easy to infer a specific relationship between attained age and mortality improvement from these observations. We do not know the exact reasons behind the recent mortality improvement, but previous studies have cited weather or environmental changes, advances in medical science, and so on, as the reasons. We also cannot conjecture a quantitative mortality improvement for the future. We can only make the observation that mortality rate changes have been experienced in recent years.

〈Table 1〉 Accumulated Improvement Rate for Estimated Male Mortality Rate

(unit: %)

Year \ age	40~44	45~49	50~54	55~59	60~64	65~69	70~74	75~79
2008	95.1	95.6	95.8	95.6	95.6	96.0	96.6	97.4
2009	90.5	91.5	91.8	91.5	91.4	92.2	93.4	94.9
2010	86.2	87.6	88.0	87.5	87.4	88.6	90.3	92.5
2011	82.1	83.8	84.3	83.8	83.6	85.1	87.4	90.2
2012	78.2	80.2	80.8	80.2	80.1	81.8	84.6	88.0
2013	74.5	76.9	77.5	76.8	76.7	78.7	81.9	85.8
2014	71.1	73.7	74.4	73.6	73.4	75.7	79.3	83.8
2015	67.8	70.6	71.4	70.5	70.4	72.8	76.8	81.8
2016	64.8	67.7	68.5	67.6	67.4	70.1	74.4	79.8
2017	61.9	64.9	65.8	64.8	64.7	67.5	72.2	78.0
2018	59.1	62.3	63.2	62.2	62.0	65.0	70.0	76.2
2019	56.6	59.8	60.7	59.7	59.5	62.7	67.9	74.5
2020	54.1	57.4	58.3	57.3	57.2	60.5	65.9	72.8
2021	51.8	55.2	56.0	55.0	54.9	58.3	64.0	71.2
2022	49.7	53.1	53.9	52.9	52.8	56.3	62.2	69.6
2023	47.6	51.0	51.8	50.8	50.7	54.4	60.5	68.2
2024	45.7	49.1	49.9	48.9	48.8	52.5	58.8	66.7
2025	43.9	47.2	48.0	47.0	47.0	50.8	57.2	65.3
2026	42.2	45.5	46.3	45.3	45.2	49.1	55.7	64.0
2027	40.5	43.8	44.6	43.6	43.6	47.5	54.2	62.7

Another recent example regarding mortality rate risks is illustrated in a study by Chen and Zhu(2007). Using the China life insurance mortality tables for 1990-1993 and 2000-2003, they discussed how to measure the impact of mortality rate risks when pricing life insurance policies. They applied actuarial methods to describe the impact of mortality rate risks on insurance product pricing. They also observed the improved mortality trends in China and showed that the premiums of term life insurances and the whole life insurances have decreased, but the premiums of annuities have increased as a consequence of the mortality rate improvements. Term life insurance policies have the largest percentage changes in premiums, followed by annuities, while whole life insurances have the smallest percentage changes in premiums. Chen and Zhu suggested natural hedging opportunities for mortality rate risks by observing inverse performances in annuity and life insurance pricing. Cox and Lin(2007) also reported that natural hedging stabilizes aggregate liability cash flows because the values of life insurance and annuity liabilities move in opposite directions in response to a change in the underlying mortality. In this paper, we provide analytical solutions for mortality rate risks.

II. Sensitivity Test and Hedge Ratio

Mortality rate change is a recent trend, especially for middle-aged males, and they should be investigated by actuaries for the mortality rate risk management. We define the mortality rate risks as follows:

Definition 1 The mortality rate risk is the risk that the actual claims associated with death are more frequent than the anticipated claims, resulting in unexpected losses.

In this section, we assume that the force of mortality may increase for specific ages unexpectedly, and that mortality rate risks may exist. If the life table constructed from a force of mortality model predicts mortality rates lower than those actually experienced by the life insurance policy-holders, then the company will face losses in the future. We consider the changes in mortality rates, which are called mortality rate shocks, and investigate the premium differences when mortality rate shocks exist.

For illustration purposes, we use Gompertz's model,

$$\mu(x) = \mu(0)c^x. \tag{1}$$

The survival function, $s(x)$, is calculated from this force of mortality function,

$$s(x) = \exp[-\int_0^x (\mu(0)c^s)ds] = \exp[-m(c^x - 1)], \tag{2}$$

where $m = \mu(0)/\log c$.

For illustration purposes, we assume that the parameter c in the force of mortality function is increased by 1%, i.e., c is changed to $1.01c$. Using this assumption, we construct the changed life table, calculate the premiums of 10-year term insurance and 10-year pure endowment, examine the gains or losses of premiums, and compute the hedge ratios.

The premiums of 10-year term insurance, $A^{\,1}_{x:\overline{10|}}$, currently have increased after the mortality rate shock, and the premiums of 10-year pure endowment, $A_{x:\overset{\,1}{\overline{10|}}}$, have decreased after the mortality rate shock[1]. Table 2 shows the changes in the premiums.

To determine the hedge ratio, let us denote the premium of 10-year term insurance after mortality rate shock by $\widetilde{A}^{\,1}_{x:\overline{10|}}$ and denote the premium of 10-year pure endowment after mortality rate shock by $\widetilde{A}_{x:\overset{\,1}{\overline{10|}}}$. The premium loss amount from 10-year term insurance is

1) Thorough out this paper, we attempt to follow the general rules for symbols of actuarial functions. See the Appendix 4 in Bowers et al(1997).

$$\text{Loss} = \widetilde{A}^{\,1}_{x:\overline{10|}} - A^{\,1}_{x:\overline{10|}} > 0 \ ,$$

while the premium gain from 10-year pure endowment is

$$\text{Gain} = A_{x:\overset{\,1}{\overline{10|}}} - \widetilde{A}_{x:\overset{\,1}{\overline{10|}}} > 0.$$

To offset the losses from the 10-year term insurance sales, we will have to sell 10-year pure endowment at the same time; that is, it is better for insurance companies to sell 10-year endowment insurance policies than to sell only 10-year term insurance policies. They can then hedge the mortality rate risk from the increasing mortality rate shock. The hedge ratio, R_x, is defined to be the face amount of the 10-year pure endowment to be sold to offset the losses from the 10-year term insurance of face amount 1 payable at the end of the year when (x) dies. The hedge ratio R_x is such that

$$\widetilde{A}^{\,1}_{x:\overline{10|}} - A^{\,1}_{x:\overline{10|}} = R_x\left(A_{x:\overset{\,1}{\overline{10|}}} - \widetilde{A}_{x:\overset{\,1}{\overline{10|}}}\right),$$

and we have

$$R_x = \frac{\widetilde{A}^{\,1}_{x:\overline{10|}} - A^{\,1}_{x:\overline{10|}}}{A_{x:\overset{\,1}{\overline{10|}}} - \widetilde{A}_{x:\overset{\,1}{\overline{10|}}}}, \qquad (3)$$

〈Table 2〉 Changes in Premiums due to Mortality Rate Shock

Change in Premiums					
Before Shock			After Shock		
Age(x)	Term Life	Pure Endowment	Age(x)	Term Life	Pure Endowment
35	0.02183	0.59598	35	0.03239	0.58722
40	0.03299	0.58684	40	0.05119	0.57178
45	0.04967	0.57318	45	0.08031	0.54791
50	0.07438	0.55301	50	0.12460	0.51180
55	0.11047	0.52366	55	0.18995	0.45891
60	0.16204	0.48195	60	0.28192	0.38544

〈Table 3〉 Premium Gains and Losses from Mortality Rate Shock

Age(x)	Losses from Term Life	Gains from Pure Endowment	R_x(Hedge Ratio)
35	0.01056	0.00876	1.20548
40	0.01820	0.01506	1.20845
45	0.03064	0.02527	1.21250
50	0.05022	0.04121	1.21864
55	0.07948	0.06475	1.22749
60	0.11988	0.09651	1.24215

Table 3 shows the gains and losses from mortality rate shock and the hedge ratio R_x. We assume that $\mu(0) = 0.0001$, $c = 0.40987$, and the annual interest rate is $i = 5\%$.

III. Hedging Strategy for Mortality Rate Risks in Developing Endowment Policies

We assume that the mortality rate shock follows a particular movement and the parameter c in the force of mortality function is increased by 1%. This assumption is just for illustration purpose and is not realistic. We cannot predict the exact amount of change in the mortality rate shock, even though we attempt to observe the trends of mortality improvement as precisely as possible. We generalize this assumption.

In this section, we do not assume the amount of mortality rate shock; we simply assume that mortality rate shocks may exist. For a given life table, the premium of n-year term life insurance with a face amount of 1 payable at the end of the year when (x) dies is

$$A^{\;1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1} \frac{d_{x+k}}{l_x}, \tag{4}$$

where $v = 1/(1+i)$, i is the annual effective interest rate, d_{x+k} is the expected the number of deaths between age $x+k$ and $x+k+1$, and l_x is the number of policy holders at age x.

Let us assume that the force of mortality μ_{x+t} at age $x+t$ is increased to μ^{ϵ}_{x+t}, by $\epsilon(x,t) > 0$,

$$\mu_{x+t} \rightarrow \mu^{\epsilon}_{x+t} = \mu_{x+t} + \epsilon(x,t),\ t \geq 0. \tag{5}$$

Then the survival probability ${}_tp_x$ will be changed to ${}_tp^{\epsilon}_x$,

$$\begin{aligned} {}_tp^{\epsilon}_x &= \exp\left(-\int_0^t (\mu_{x+s} + \epsilon(x,s))ds\right) \\ &= \exp\left(-\int_0^t \mu_{x+s}ds\right)\exp\left(-\int_0^t \epsilon(x,s)ds\right) \\ &= {}_tp_x \exp\left(-\int_0^t \epsilon(x,s)ds\right) < {}_tp_x. \end{aligned} \tag{6}$$

The premium of term insurance after mortality rate shock $\epsilon(x,t)$ is

$$\widetilde{A}^{1}_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\frac{\widetilde{d}_{x+k}}{l_x} > \sum_{k=0}^{n-1} v^{k+1}\frac{d_{x+k}}{l_x} = A^{1}_{x:\overline{n|}}. \tag{7}$$

The amount of loss from term insurance after the mortality rate shock is

$$Loss_x = \widetilde{A}^{1}_{x:\overline{n|}} - A^{1}_{x:\overline{n|}} > 0. \tag{8}$$

As a hedging strategy, we consider n-year pure endowment to offset the losses from n-year term insurance. The net single premium of n-year pure endowment issued to (x) before mortality rate shock is

$$A_{x:\overset{1}{\overline{n|}}} = {}_nE_x = v^n\ {}_np_x. \tag{9}$$

After the force of mortality μ_{x+t} is increased to μ^{ϵ}_{x+t} by $\epsilon(x,t) \rangle\ 0$, the net single premium of the n-year pure endowment will be decreased to $\widetilde{A}_{x:\overset{1}{\overline{n|}}}$. The amount of gain from the n-year pure endowment after the mortality rate shock is

$$Gain_x = A_{x:\overline{n|}}^{\ \ 1} - \widetilde{A}_{x:\overline{n|}}^{\ \ 1} = {}_nE_x - {}_n\widetilde{E}_x > 0. \tag{10}$$

We want to offset the losses from n-year term insurance with the gains from n-year pure endowment. The hedge ratio, R_x, is the ratio of the n-year pure endowment required to offset the losses from n-year term insurance in n-year endowment policies.

Note that the ratio between term insurance and pure endowment is 1 in traditional endowments. Here, we develop a new type of endowment, called modified endowments, that pay 1 when the insured (x) dies in n years or pay R_x when the insured survives at age $x+n$.

Definition 2 The hedge ratio R_x for age x is the ratio of the n-year pure endowment required to offset the losses from n-year term insurance in a modified endowment, i.e., R_x is the number such that

$$Loss_x = \widetilde{A}_{x:\overline{n|}}^{1} - A_{x:\overline{n|}}^{1} = R_x({}_nE_x - {}_n\widetilde{E}_x) = Gain_x. \tag{11}$$

We calculate R_x for each age x. From the definition of R_x, we have

$$\widetilde{A}_{x:\overline{n|}}^{1} + R_x\,{}_n\widetilde{E}_x = A_{x:\overline{n|}}^{1} + R_x\,{}_nE_x. \tag{12}$$

For convenience, let us denote the liability of a modified endowment before mortality rate shock to be L_x and the liability after mortality rate shock to be $\widetilde{L}_x$,

$$L_x = A_{x:\overline{n|}}^{1} + R_x\,{}_nE_x, \tag{13}$$

and

$$\widetilde{L}_x = \widetilde{A}_{x:\overline{n|}}^{1} + R_x\,{}_n\widetilde{E}_x. \tag{14}$$

We want to find the hedge ratio R_x such that

$$L_x = A^{1}_{x:\overline{n}|} + R_{x\,n}E_x = \widetilde{L}_x = \widetilde{A}^{1}_{x:\overline{n}|} + R_{x\,n}\widetilde{E}_x. \tag{15}$$

Note that the premium of a traditional n-year endowment is

$$\begin{aligned} A_{x:\overline{n}|} &= A^{1}_{x:\overline{n}|} + {}_nE_x \\ &= 1 - d\ddot{a}_{x:\overline{n}|}, \end{aligned}$$

where $\ddot{a}_{x:\overline{n}|}$ is the premium of the n-year temporary annuity-due.

The liability L_x of a modified endowment before the mortality rate shock is

$$\begin{aligned} L_x &= A^{1}_{x:\overline{n}|} + R_{x\,n}E_x \\ &= A_{x:\overline{n}|} - {}_nE_x + R_{x\,n}E_x \\ &= A_{x:\overline{n}|} + (R_x - 1)_nE_x \\ &= 1 - d\ddot{a}_{x:\overline{n}|} + (R_x - 1)_nE_x \\ &= 1 - d\sum_{k=0}^{n-1} v^k\,{}_kp_x + (R_x - 1)v^n\,{}_np_x. \end{aligned} \tag{16}$$

By the same method, the liability $\widetilde{L}_x$ after the mortality rate shock is

$$\widetilde{L}_x = 1 - d\sum_{k=0}^{n-1} v^k\,{}_k\widetilde{p}_x + (R_x - 1)v^n\,{}_n\widetilde{p}_x. \tag{17}$$

The difference ΔL_x between the liabilities L_x and $\widetilde{L}_x$ is

$$\begin{aligned} \Delta L_x &= \widetilde{L}_x - L_x \\ &= -d\sum_{k=0}^{n-1} v^k\left({}_k\widetilde{p}_x - {}_kp_x\right) + (R_x - 1)v^n\left({}_n\widetilde{p}_x - {}_np_x\right). \end{aligned} \tag{18}$$

We determine R_x such that the difference between the liabilities, ΔL_x, is as small as possible,

$$\Delta L_x = \widetilde{L}_x - L_x = -d\sum_{k=0}^{n-1} v^k \left({}_k\widetilde{p}_x - {}_kp_x\right) + (R_x - 1)v^n \left({}_n\widetilde{p}_x - {}_np_x\right) = 0.$$

Let us analyze the difference $\Delta L_x = \widetilde{L}_x - L_x$ between the liabilities.

$$\begin{aligned}\Delta L_x = \widetilde{L}_x - L_x &= -d\sum_{k=0}^{n-1} v^k \left({}_k\widetilde{p}_x - {}_kp_x\right) + (R_x - 1)v^n \left({}_n\widetilde{p}_x - {}_np_x\right) \\ &= -d\sum_{k=0}^{n-1} v^k \frac{{}_k\widetilde{p}_x - {}_kp_x}{{}_kp_x} {}_kp_x + (R_x - 1)v^n \frac{{}_n\widetilde{p}_x - {}_np_x}{{}_np_x} {}_np_x .\end{aligned}$$

We define the function as

$$f(k) = \frac{{}_k\widetilde{p}_x - {}_kp_x}{{}_kp_x}. \tag{19}$$

Note that $f(0) = (1-1)/1 = 0$. The difference between the liabilities is

$$\Delta L_x = -d\sum_{k=0}^{n-1} v^k f(k)\,{}_kp_x + (R_x - 1)v^n f(n)\,{}_np_x \tag{20}$$

If we assume that the function $f(k)$ is twice differentiable, then, by Taylor's formula with an integral remainder, the function can be expressed as

$$\begin{aligned}f(k) &= f(0) + f'(0)k + \int_0^k (k-w)f''(w)dw \\ &= f'(0)k + \int_0^k (k-w)f''(w)dw.\end{aligned}$$

The difference between the liabilities then becomes

$$\begin{aligned}\Delta L_x &= -d\sum_{k=0}^{n-1} v^k f'(0)k\,{}_kp_x - d\sum_{k=0}^{n-1}\left[v^k\,{}_kp_x \int_0^k (k-w)f''(w)dw\right] \\ &\quad + (R_x - 1)v^n n f'(0)\,{}_np_x + (R_x - 1)v^n\,{}_np_x \int_0^n (n-w)f''(w)dw\end{aligned}$$

$$= f'(0)\left[-d\sum_{k=0}^{n-1} v^k k\,{}_kp_x + (R_x - 1)v^n n\,{}_np_x\right]$$
$$-d\sum_{k=0}^{n-1}\left[v^k\,{}_kp_x\int_0^k (k-w)f''(w)dw\right] + (R_x - 1)v^n\,{}_np_x\int_0^n (n-w)f''(w)dw$$
$$= \text{I} + \text{II},$$

where

$$\text{I} = f'(0)\left[-d\sum_{k=0}^{n-1} v^k k\,{}_kp_x + (R_x - 1)v^n n\,{}_np_x\right],$$

and

$$\text{II} = -d\sum_{k=0}^{n-1}\left[v^k\,{}_kp_x\int_0^k (k-w)f''(w)dw\right] + (R_x - 1)v^n\,{}_np_x\int_0^n (n-w)f''(w)dw.$$

Now, let us find R_x such that the difference ΔL_x between the liabilities is minimized,

$$\Delta L_x = \text{I} + \text{II} \approx 0.$$

One strategy is to find R_x such that the first term I of ΔL_x equals 0,

$$\text{I} = f'(0)\left[-d\sum_{k=0}^{n-1} v^k k\,{}_kp_x + (R_x - 1)v^n n\,{}_np_x\right] = 0, \tag{21}$$

and the second term II of ΔL_x becomes nearly 0,

$$\text{II} = -d\sum_{k=0}^{n-1}\left[v^k\,{}_kp_x\int_0^k (k-w)f''(w)dw\right] + (R_x - 1)v^n\,{}_np_x\int_0^n (n-w)f''(w)dw \approx 0.$$

From the equation(21), we obtain the following:

$$-d\sum_{k=0}^{n-1} v^k k\,{}_kp_x + (R_x - 1)v^n n\,{}_np_x = 0 \tag{22}$$

$$\Leftrightarrow -d\left[(I\ddot{a})_{x:\overline{n|}} - \ddot{a}_{x:\overline{n|}}\right] + (R_x - 1)n\,{}_nE_x = 0 \tag{23}$$

$$\Leftrightarrow -d(Ia)_{x:\overline{n-1|}} + (R_x - 1)n\,{}_nE_x = 0 \tag{24}$$

$$\Leftrightarrow (R_x - 1)n\,{}_nE_x = d(Ia)_{x:\overline{n-1|}}, \tag{25}$$

where

$$(Ia)_{x:\overline{n-1|}} = \sum_{k=0}^{n-1} v^k k\,{}_kp_x.$$

Therefore, we have the following theorem.

Theorem 1 The hedge ratio R_x of the n-year pure endowment for age x required to offset the losses from the sale of 1 unit of n-year term insurance(with face amount 1) is approximately

$$R_x = 1 + \frac{d}{n}\frac{(Ia)_{x:\overline{n-1|}}}{{}_nE_x}. \tag{26}$$

Remark 1 The hedge ratio R_x in theorem 1 is independent of the amount of mortality rate shock $\epsilon(x,t)$. Even though it is greater than 1 for any age x, this may not apply for some ages, and we will later solve this problem using a stochastic mortality rate model and Malliavin calculus. Malliavin calculus is summarized at Appendix.

We can interpret Theorem 1 using the sensitivity of the liabilities with respect to the change in mortality rates with the following assumption. For the survival probability, ${}_kp_x = \exp\left(-\int_0^k \mu_{x+s}ds\right)$, we assume that the sensitivity of the liability with respect to the change of mortality rates is approximately

$$\frac{\partial_k p_x}{\partial \mu} \approx -k\exp\left(-\int_0^k \mu_{x+s}\,ds\right) = -k\,{}_kp_x. \tag{27}$$

Theorem 2 With the above assumption, the hedge ratio R_x of the n-year pure endowment for age x required to offset the losses from n-year term insurance is the number that satisfies

$$\frac{\partial L_x}{\partial \mu} = 0. \tag{28}$$

Proof From(16) we have

$$\begin{aligned} L_x &= A^{\;1}_{x:\overline{n|}} + R_{x\;n}E_x \\ &= 1 - d\sum_{k=0}^{n-1} v^k\;{}_kp_x + (R_x - 1)v^n\;{}_np_x. \end{aligned}$$

We also have the following equivalences:

$$\begin{aligned} &\frac{\partial L_x}{\partial \mu} = 0 \\ &\Leftrightarrow -d\sum_{k=0}^{n-1} v^k \frac{\partial_k p_x}{\partial \mu} + (R_x - 1)v^n \frac{\partial_n p_x}{\partial \mu} = 0 \\ &\Leftrightarrow d\sum_{k=0}^{n-1} v^k k\;{}_kp_x - (R_x - 1)v^n n\;{}_np_x = 0 \\ &\qquad\qquad \Leftrightarrow (R_x - 1)n\;{}_nE_x = d(Ia)_{x:\overline{n-1|}} \\ &\Leftrightarrow R_x = 1 + \frac{d}{n}\frac{(Ia)_{x:\overline{n-1|}}}{{}_nE_x}. \end{aligned}$$

Remark 2 If the force of mortality $\mu_x = \mu$, a constant, then we have

$${}_kp_x = \exp\left(-\int_0^k \mu_{x+s}ds\right) = e^{-\mu k}. \tag{29}$$

From(16) we have

$$L_x = A^{\;1}_{x:\overline{n|}} + R_{x\;n}E_x = 1 - d\sum_{k=0}^{n-1} v^k\;{}_kp_x + (R_x - 1)v^n\;{}_np_x.$$

Now we consider L_x as a function of both mortality rate μ and interest rate i.

Then, we have the following equivalences on the change of the liabilities L_x with respect to the change of mortality rate μ,

$$\frac{\partial L_x}{\partial \mu} = 0 \tag{30}$$

$$\Leftrightarrow \frac{\partial \left\{ 1 - d \sum_{k=0}^{n-1} v^k \, {}_k p_x + (R_x - 1) v^n \, {}_n p_x \right\}}{\partial \mu} = 0$$

$$\Leftrightarrow - d \sum_{k}^{n-1} v^k \frac{\partial \, {}_k p_x}{\partial \mu} + (R_x - 1) v^n \frac{\partial \, {}_n p_x}{\partial \mu} = 0$$

$$\Leftrightarrow d \sum_{k}^{n-1} v^k k \, {}_k p_x - (R_x - 1) v^n n \, {}_n p_x = 0$$

$$\Leftrightarrow (R_x - 1) n \, {}_n E_x = d (Ia)_{x : \overline{n-1|}}$$

$$\Leftrightarrow R_x = 1 + \frac{d}{n} \frac{(Ia)_{x : \overline{n-1|}}}{{}_n E_x} .$$

Recall that the difference ΔL_x between the liabilities is

$$\Delta L_x = \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = f'(0) \left[- d \sum_{k=0}^{n-1} v^k k \, {}_k p_x + (R_x - 1) v^n n \, {}_n p_x \right],$$

and

$$\mathrm{II} = - d \sum_{k=0}^{n-1} \left[v^k \, {}_k p_x \int_0^k (k - w) f''(w) dw \right] + (R_x - 1) v^n \, {}_n p_x \int_0^n (n - w) f''(w) dw$$

If we set R_x such that

$$R_x = 1 + \frac{d}{n} \frac{(Ia)_{x : \overline{n-1|}}}{{}_n E_x},$$

then the first term I of ΔL_x equals 0,

$$\mathrm{I} = f'(0)\left[-d\sum_{k=0}^{n-1} v^k k\,{}_kp_x + (R_x - 1)v^n n\,{}_np_x\right] = 0,$$

and

$$\Delta L_x = \mathrm{II}.$$

Therefore, the hedging strategy with $R_x = 1 + \frac{d}{n}\frac{(Ia)_{x:\overline{n-1|}}}{{}_nE_x}$ does not guarantee a perfect hedging. Mortality rate risks still remain since $\Delta L_x = \mathrm{II}$ may not be equal to 0.

Definition 3 The residual risk RR_x for age x is the amount $\Delta L_x = \mathrm{II}$ when the hedging strategy uses $R_x = 1 + \frac{d}{n}\frac{(Ia)_{x:\overline{n-1|}}}{{}_nE_x}$ (i.e. $\mathrm{I} = 0$).

Now, let us analyze the residual risk.

$$RR_x = \Delta L_x = \mathrm{II}$$

$$= -d\sum_{k=0}^{n-1}\left[v^k\,{}_kp_x\int_0^k (k-w)f''(w)dw\right] + (R_x-1)v^n\,{}_np_x\int_0^n (n-w)f''(w)dw \qquad (31)$$

$$= -d\sum_{k=0}^{n-1}\left[v^k\,{}_kp_x(f(k) - kf'(0))\right] + (R_x - 1)v^n\,{}_np_x(f(n) - nf'(0))$$

$$= -d\sum_{k=0}^{n-1}\left[v^k\,{}_kp_x(f(k) - kf'(0))\right] + \frac{d}{n}(Ia)_{x:\overline{n-1|}}(f(n) - nf'(0))$$

$$= -d\sum_{k=0}^{n-1} v^k\,{}_kp_x f(k) + df'(0)\sum_{k=0}^{n-1} kv^k\,{}_kp_x + \frac{d}{n}(Ia)_{x:\overline{n-1|}}(f(n) - nf'(0))$$

$$= -d\sum_{k=0}^{n-1} v^k\,{}_kp_x f(k) + df'(0)(Ia)_{x:\overline{n-1|}} + \frac{d}{n}(Ia)_{x:\overline{n-1|}}(f(n) - nf'(0))$$

$$= -d\sum_{k=0}^{n-1} v^k\,{}_kp_x f(k) + d(Ia)_{x:\overline{n-1|}}\frac{f(n)}{n}$$

$$= d\left[(Ia)_{x:\overline{n-1|}}\frac{f(n)}{n} - \sum_{k=0}^{n-1} v^k\,{}_kp_x f(k)\right]. \qquad (32)$$

Since we cannot exactly determine the function $f(k)$,(32) does not provide us the risk RR_x. However, we have another way to estimate RR_x if we assume that RR_x is a measure of the difference between a non-perfect hedging strategy and a perfect hedging strategy. If we set up the liability L_x as follows

$$L_x = A^{\ 1}_{x\,:\,\overline{n|}} + {}_nE_x + d\ddot{a}_{x\,:\,\overline{n|}}, \tag{33}$$

then $\Delta L_x = 0$, i.e., a perfect hedging strategy. For the perfect hedging strategy, the liability $\tilde{L}_x$ after the mortality rate shock is

$$\begin{aligned}\tilde{L}_x &= L_x + \Delta L_x \\ &= A^{\ 1}_{x\,:\,\overline{n|}} + {}_nE_x + d\ddot{a}_{x\,:\,\overline{n|}}.\end{aligned}$$

For a non-perfect hedging strategy, the liability after the mortality rate shock is

$$\begin{aligned}\tilde{L}_x &= L_x + \Delta L_x \\ &= L_x + \mathrm{I} + \mathrm{II} \\ &= A^{\ 1}_{x\,:\,\overline{n|}} + R_x\,{}_nE_x + RR_x\end{aligned}$$

If we set

$$R_x\,{}_nE_x + RR_x = {}_nE_x + d\ddot{a}_{x\,:\,\overline{n|}},$$

then the residual risk is

$$\begin{aligned}RR_x &= {}_nE_x + d\ddot{a}_{x\,:\,\overline{n|}} - R_x\,{}_nE_x \\ &= (1 - R_x)\,{}_nE_x + d\ddot{a}_{x\,:\,\overline{n|}} \\ &= -\frac{d}{n}\frac{(Ia)_{x\,:\,\overline{n-1|}}}{{}_nE_x}\,{}_nE_x + d\ddot{a}_{x\,:\,\overline{n|}}, \ \text{by(26)} \\ &= -\frac{d}{n}(Ia)_{x\,:\,\overline{n-1|}} + d\ddot{a}_{x\,:\,\overline{n|}} \\ &= d\left[\ddot{a}_{x\,:\,\overline{n|}} - \frac{1}{n}(Ia)_{x\,:\,\overline{n-1|}}\right]\end{aligned}$$

$$= \frac{d}{n}(D\ddot{a})_{x:\overline{n}|},$$

where $(D\ddot{a})_{x:\overline{n}|} = \sum_{k=0}^{n-1}(n-k)v^k\,{}_kp_x$.

We now have the following theorem.

Theorem 3 Let us assume that RR_x is a measure of the difference between the non-perfect hedging strategy with the hedge ratio $R_x = 1 + \frac{d}{n}\frac{(Ia)_{x:\overline{n-1}|}}{{}_nE_x}$ and the perfect hedging strategy(33). Under these conditions, the residual risk of the non-perfect hedging strategy is

$$RR_x = \frac{d}{n}(D\ddot{a})_{x:\overline{n}|}. \tag{34}$$

Remark 3 Let us find a perfect hedge ratio, $R_x^{perfect}$, of n-year pure endowment. Comparing

$$L_x = A^{\,1}_{x:\overline{n}|} + {}_nE_x + d\ddot{a}_{x:\overline{n}|}$$

and

$$L_x = A^{\,1}_{x:\overline{n}|} + R_x^{perfect}\,{}_nE_x$$

we have

$$R_x^{perfect}\,{}_nE_x = {}_nE_x + d\,\ddot{a}_{x:\overline{n}|}.$$

Therefore, a perfect hedge ratio of the n-year pure endowment to hedge the mortality rate risks in an n-year endowment[2)] is

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2) Note that a perfect hedging strategy is $L_x = A^{\,1}_{x:\overline{n}|} + {}_nE_x + d\ddot{a}_{x:\overline{n}|} = 1$. So $R_x^{perfect}$ is the risk measure only when $L_x = 1$ which is a special case of perfect hedging. We want to find more general hedging strategies.

$$R_x^{perfect} = 1 + d\frac{\ddot{a}_{x:\overline{n}|}}{{}_nE_x}$$

$$= 1 + d\ddot{s}_{x:\overline{n}|}. \qquad (35)$$

〈Table 4〉 Hedge Ratios and Residual Risks

Age(x)	R_x (1% increase in c)	R_x	RR_x	$R_x^{perfect}$
35	1.20548	1.18792	0.22639	1.64127
40	1.20845	1.19507	0.22564	1.64784
45	1.21250	1.20386	0.22452	1.65799
50	1.21864	1.21480	0.22283	1.67379
55	1.22749	1.22867	0.22030	1.69870
60	1.24215	1.24689	0.21656	1.73868

Table 4 shows the hedge ratios R_x and residual risks RR_x when we assume that the force of mortality follows Gompertz's model. In the second column of Table 4, we assume that the parameter c in the force of mortality function is increased by 1%. In the third column, we calculate the hedge ratios according to Theorem 1. In the fourth column, we determine the residual risks. We calculate a perfect hedge ratio in the last column.

We see in Table 4 that the hedge ratios R_x and the perfect hedge ratio $R_x^{perfect}$ are all greater than 1. However, this may not apply for the contracts with low mortality rate risks for young ages. We will solve this problem using a stochastic mortality rate model and Malliavin calculus in the following sections.

IV. Stochastic Force of Mortality using the Brownian Gompertz Model

Until now, we assume that mortality rate shocks exist and attempt to find the hedge ratio, R_x, which is independent of the amount of mortality rate shocks. However, the hedging strategy is not perfect, and residual risks remain because the approximation using Taylor's formula may have error terms. To improve the hedging strategies, we can consider stochastic mortality rate models and calculate the sensitivities of liabilities directly using Malliavin calculus.

There are several families of deterministic analytical laws of mortality such as De Moivre, Gompertz, Makeham, and Weibull. The Gompertz law of mortality is given by

$$\mu(x) = \mu(0)c^x \tag{36}$$

where $\mu(0) > 0, c > 1, x \geq 0$.

For the stochastic mortality rate models, we refer to Lee and Carter(1992), Lee(2000), Yang(2001), Dahl(2004), Biffis(2005), Milevsky and Promislow(2001)[1], Woodbury and Manton(1997), Renshaw and Haberman(2000), and Yashin et al.(1985).

For illustration purpose, we consider a stochastic law of mortality, especially the Brownian Gompertz(BG) model, which is based on Milevsky and Promislow(2001). The BG force of the mortality process is expected to grow exponentially; the variance is proportional to the value of the hazard rate, and this process never becomes zero. In this paper, we consider a simple stochastic force of mortality process, $\{\mu(t) : t \geq 0\}$, as follows[2]

1) The choice and justification of stochastic mortality rate models may depend on the data and insurance policies.

2) Milevsky and Promislow(2001) suggests the model, $\mu(t,x) = \zeta \exp(\xi x + \sigma Z(t))$, for the force of mortality. Here, we assume that age $x = 0$ for illustration purpose.

$$\mu(t) = \mu(0)\exp(\sigma Z(t)), \tag{37}$$

where $\sigma > 0, \mu(0) > 0$, and the dynamics of $\{Z(t) : t \geq 0\}$ are described by the stochastic differential equation,

$$dZ(t) = -bZ(t)dt + dW(t), \tag{38}$$

with $Z(0) = 0, b \geq 0$, and $W(t)$ is the standard Brownian motion. Note that if $b = 0$, the process $\mu(t)$ is the geometric Brownian motion.

We can solve(38) for $Z(t)$,

$$Z(t) = \int_0^t e^{-b(t-s)} dW(s). \tag{39}$$

The mean value of $Z(t)$ is

$$E(Z(t)) = 0, \tag{40}$$

and the variance of $Z(t)$ is

$$Var(Z(t)) = E(Z(t)^2) = \int_0^t e^{-2b(t-s)} dW(s) = \frac{1-e^{-2bt}}{2b}. \tag{41}$$

Note that $Var(Z(t)) < t$, so the process has a smaller variance than $W(t)$, and $Var(Z(t))$ converges to t as b goes to 0.

The expected value of the stochastic force of mortality $\mu(t)$ is

$$E(\mu(t)) = \mu(0)\exp\left\{\frac{\sigma^2}{2}\left(\frac{1-e^{-2bt}}{2b}\right)\right\}. \tag{42}$$

Note that $E(\mu(t))$ converges to the Gompertz law of mortality as b goes to 0,

$$E(\mu(t)) \rightarrow \mu(0)\exp\left(\frac{\sigma^2}{2}t\right), \text{ as } b \rightarrow 0. \tag{43}$$

Let us consider the dynamics of $\{\mu(t):t\geq 0\}$. Using Itô's lemma, we have

$$d\mu(t)=\sigma\mu(t)dZ(t)+\frac{1}{2}\sigma^2\mu(t)dt \tag{44}$$

$$=\sigma\mu(t)\{-bZ(t)dt+dW(t)\}+\frac{1}{2}\sigma^2\mu(t)dt,\ \text{by}(38), \tag{45}$$

$$=\sigma\mu(t)\left\{-b\frac{1}{\sigma}ln\left(\frac{\mu(t)}{\mu(0)}\right)dt+dW(t)\right\}+\frac{1}{2}\sigma^2\mu(t)dt,\ \text{by}(37), \tag{46}$$

$$=\left\{\frac{1}{2}\sigma^2+b\ln\mu(0)-b\ln\mu(t)\right\}\mu(t)dt+\sigma\mu(t)dW(t) \tag{47}$$

We will use this formula to determine a hedging strategy using Malliavin calculus in the Appendix.

For $T>0$, we assume that $F=X(T)$ is a solution to the stochastic differential equation

$$dX(t)=\beta(X(t))dt+\sigma(X(t))dW(t).$$

We define the tangent process $\{Y(t)=\frac{\partial X(t)}{\partial X(0)}:t\geq 0\}$ of $X(t)$ as the associated first variation process defined by the stochastic differential equation

$$dY(t)=\beta'(X(t))Y(t)dt+\sigma'(X(t))Y(t)dW(t),\quad Y(0)=1 \tag{48}$$

where primes denote derivatives.

Then, the Malliavin derivative of $F=X(T)$ is given by

$$D_sX(T)=\sigma(X(s))Y(s)^{-1}Y(T) \tag{49}$$

for $s\leq T$, and is zero otherwise.

For $X(0)=x$ and $F=X(T)$, we calculate the price sensitivity with respect to x

$$\frac{\partial}{\partial x}E[\psi(X(t))]=E[\psi'(X(T))Y(T)]. \tag{50}$$

Using Malliavin calculus, we attempt to determine the weight $\pi = \delta(u)$ for some adapted process u, such that

$$\frac{\partial}{\partial x}E[\psi(X(t))] = E[\psi(X(T))\pi]$$
$$= E[\psi(X(T))\delta(u)]$$
$$= E[\langle D\psi(X(T)), u\rangle], \text{ by(83)}$$
$$= E[\psi'(X(T))\langle DX(T), u\rangle], \text{ by(85)}$$
$$= E\left[\psi'(X(T))\int_0^T (D_t X(T))u(t)dt\right], \text{ by(81)}$$
$$= E\left[\psi'(X(T))\,Y(T)\int_0^T Y(T)^{-1}\sigma(X(T))u(t)dt\right], \text{ by(49)} \qquad (51)$$

Comparing(50) and(51), we need

$$\int_0^T Y(t)^{-1}\sigma(X(t))u(t)dt = 1, \qquad (52)$$

to produce the solution

$$u(t) = \frac{Y(t)}{T\sigma(X(t))}. \qquad (53)$$

Therefore, we can find the weight $\pi = \delta(u)$ when $F = X(T)$

$$\pi = \delta(u) = \int_0^T \frac{Y(t)}{T\sigma(X(t))}dW(t), \text{ by(84)} \qquad (54)$$

In a similar manner, we can find a weight $\pi = \delta(u)$ when F is the mean value of the process $\{X(t): 0 \leq t \leq T\}$

$$F = \int_0^T X(t)dt. \qquad (55)$$

A weight $\pi = \delta(u)$ is given by

$$\pi = \delta(u) = \delta\left(\frac{2Y^2(t)}{\sigma(X(t))}\left(\int_0^T Y(s)ds\right)^{-1}\right). \tag{56}$$

V. A Hedging Strategy using the BG Stochastic Mortality Model and Malliavin Calculus

Now, we begin with the sale of term insurance. Under a stochastic force of mortality model, the net single premium of n-year term life insurance with a face amount of 1 payable at the end of the year when (x) dies is

$$A^{1}_{x:\overline{n}|} = E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}{}_{t}p_x(\omega)q_{x+t}(\omega)\right], \tag{57}$$

where $v = 1/(1+i)$, i is the annual effective interest rate, Ω is the set of scenarios, and ω is a scenario in which $\omega \in \Omega$.

For each $x > 0, t \geq 0$, and $\omega \in \Omega$, the survival probability ${}_tp_x(\omega)$ is

$${}_tp_x(\omega) = \exp\left(-\int_0^t \mu_x(s,\omega)ds\right), \tag{58}$$

where $\mu_x(s,\omega)$ is the force of mortality at age $x+s$ on the scenario ω, and the death probability $q_{x+t}(\omega)$ is

$$q_{x+t}(\omega) = 1 - p_{x+t}(\omega). \tag{59}$$

We, therefore, can rewrite(57) as

$$A^{1}_{x:\overline{n}|} = E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}{}_{t}p_x(\omega) - \sum_{t=0}^{n-1} v^{t+1}{}_{t+1}p_x(\omega)\right]. \tag{60}$$

As a hedging strategy, we consider the sales of n-year pure endowments to offset any losses from the sales of n-year term insurance. The net single premium of the n-year pure endowment issued to (x) is

$$A_{x:\overset{\ \ 1}{\overline{n|}}} = {}_nE_x = E^{\Omega}\left[v^n\,{}_np_x(\omega)\right]. \tag{61}$$

We offset the losses from n-year term insurance sales with the sales of n-year pure endowments. Let us define the hedge ratio, R_x, to be the number of n-year pure endowments that need to be sold to offset the losses from the sales of n-year term insurance policies. Let us denote the liability L_x to be

$$L_x = A^{\ 1}_{x:\overline{n|}} + R_x\,{}_nE_x. \tag{62}$$

We determine the hedge ratio R_x such that the sensitivity of the liability with respect to the mortality rate changes equals 0,

$$\frac{\partial L_x}{\partial \mu} = 0, \tag{63}$$

where $\mu = \mu_x(0,\omega)$.

The liability L_x is

$$L_x = A^{\ 1}_{x:\overline{n|}} + R_x\,{}_nE_x$$
$$= E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}\,{}_tp_x(\omega) - \sum_{t=0}^{n-1} v^{t+1}\,{}_{t+1}p_x(\omega)\right] + R_x E^{\Omega}\left[v^n\,{}_np_x(\omega)\right]. \tag{64}$$

The sensitivity of the liability with respect to the mortality rate changes is

$$\frac{\partial L_x}{\partial \mu} = \frac{\partial}{\partial \mu} E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}\,{}_tp_x(\omega) - \sum_{t=0}^{n-1} v^{t+1}\,{}_{t+1}p_x(\omega)\right] + R_x \frac{\partial}{\partial \mu} E^{\Omega}\left[v^n\,{}_np_x(\omega)\right]. \tag{65}$$

For a fixed age x, we assume that $\mu_x(t,\omega) = \mu(x+t,\omega)$, and the dynamics of the force of mortality are given by[3)]

$$d\mu_x(t) = \beta(\mu_x(t))dt + \sigma(\mu_x(t))dW(t)$$
$$= \left\{\frac{1}{2}\sigma^2 + b\ln\mu_x(0) - b\ln\mu_x(t)\right\}\mu_x(t)dt + \sigma\mu_x(t)dW(t). \qquad (66)$$

The tangent process $\{Y(t) = \frac{\partial\mu_x(t)}{\partial\mu} : t \geq 0\}$ of $\mu_x(t)$ is the associated first variation process, defined by the stochastic differential equation,

$$dY(t) = \beta'(\mu_x(t))Y(t)dt + \sigma'(\mu_x(t))Y(t)dW(t), \ \ Y(0) = 1, \qquad (67)$$

where primes denote derivatives.

For a given $x > 0$, the expected value of the survival probability is

$$E^{\Omega}[{}_tp_x(\omega)] = E^{\Omega}\left[\exp\left(-\int_0^t \mu_x(s,\omega)ds\right)\right]$$
$$= E^{\Omega}[\psi(F(t,\omega))], \qquad (68)$$

where

$$F(t,\omega) = \int_0^t \mu_x(s,\omega)ds, \qquad (69)$$

and

$$\psi(F(t,\omega)) = \exp(-F(t,\omega)). \qquad (70)$$

Using the result of Malliavin calculus, we have

$$\frac{\partial}{\partial\mu}E^{\Omega}[\psi(F(t,\omega))] = E^{\Omega}[\psi((F(t,\omega))\pi(t,\omega)], \qquad (71)$$

where the weight $\pi = \delta(u)$ is given by

3) Here, we consider the dynamics of the force of mortality under the Brownian Gompertz(BG) model. The dynamics may change according to the choice of mortality rate models.

$$\pi(t,\omega) = \delta(u) = \delta\left(\frac{2Y^2(s,\omega)}{\sigma(\mu_x(s,\omega))}\left(\int_0^t Y(l,\omega)dl\right)^{-1}\right)$$

$$= \int_0^t \frac{2Y^2(s,\omega)}{\sigma(\mu_x(s,\omega))}\left(\int_0^t Y(l,\omega)dl\right)^{-1} dW(s). \tag{72}$$

The hedge ratio R_x such that the sensitivity of the liability with respect to the mortality rate changes equals 0, $\frac{\partial L_x}{\partial \mu} = 0$, is expressed as follows

$$R_x = -\frac{E^{\Omega}\left[\sum_{t=0}^{n-1} v^{t+1}\left\{{}_tp_x(\omega)\pi(t,\omega) - {}_{t+1}p_x(\omega)\pi(t+1,\omega)\right\}\right]}{E^{\Omega}\left[v^n\, {}_np_x(\omega)\pi(n,\omega)\right]}. \tag{73}$$

VI. Numerical Examples

We have conducted a simulation to calculate the hedge ratio R_x under the BG stochastic mortality model. $b = 0.5$ and $\sigma = 0.20$ are estimated such that the average survival probabilities under the stochastic mortality rate term structure are consistent with the values under the deterministic term structure model. And $\sigma = 0.23$ is arbitrarily selected in order to see the effect on hedge ratio changes(The case $\sigma < 0.20$ is not assumed because the risk actually does not increase if σ decreases). Finally, the parameter $i = 5\%$ is fixed and arbitrarily appointed in the model. We use $\mu_x(0) = E^{\Omega}[\mu(x,\omega)]$ as the initial values. For the 10-year term insurance, we show the hedge ratios, R_x,(73), of the pure endowment in Table 5.

Note that the hedge ratios in Tables 4 and 5 are different. We do not need to use any approximations, so there are not residual risks when a stochastic mortality model and Malliavin calculus are used. This is an improvement in the hedging strategies.

Due to fixed 10-year maturity and the characteristics of the mortality model, the hedge ratios are different by age group. From Table 5, the hedge ratios for young ages(35 and 40) are less than 1. We can also see that the hedge ratios grow rapidly as age increases, so we will give more benefit in modified pure endowments to hedge the mortality rate risks for old ages. It is very natural way to compensate the loss from death by the gain from survive because the death probability of old age is higher. We also noticed that the hedge ratios depend on the level of volatilities of mortality rate movements measured by σ. The death rate for younger ages decreases in 10-year period if volatility increases. However, the death rate for older ages sharply increases in 10 years. Therefore, relatively small hedge ratio is necessary for younger ages, but high hedge ratio is necessary for old ages.

The explanations on the results from data analysis in this chapter are as following:

1) More precise hedge ratios can be obtained when the BG stochastic mortality rate model is used.

2) The influence of the volatility σ should be a consideration for mortality risk management.

3) As in the deterministic model, the hedge ratio is increasing as age increases under the stochastic mortality rate model.

4) The hedge ratios for young age are less than 1 under the stochastic mortality model, while hedge ratios for all age group are bigger than 1 under the deterministic model.

Lastly, the implications of this analysis are as follows: More benefit is necessary for old age group and more precise mortality model, such as BG, is needed to hedge mortality risk. In addition, volatility must be taken into consideration because it is one of the key drivers for determining hedge ratios.

〈Table 5〉 Hedge Ratios

Age / σ	35	40	45	50	55	60
0.23	0.81378	0.90778	1.05951	1.31239	1.76686	2.72489
0.20	0.89650	0.97027	1.08998	1.29090	1.65499	2.42923

VII. Conclusion

We observed that mortality rates have experienced changes, especially for middle-aged males, over the last few decades. If mortality rate shocks exist, insurance companies may face losses from life insurance sales. As a hedging strategy, an insurance company may develop modified endowments. We presented the hedge ratios of pure endowments to offset the losses from term life insurance in developing modified endowment policies.

We attempted to find hedging strategies using three steps. First, we employed a deterministic mortality rate model, the Gompertz model, and assumed a given shock in the mortality rates to determine a hedging strategy. In the second step, we used an approximation to determine a hedging strategy under any unknown mortality rate shock. We derived a non-perfect hedging strategy with residual risks. Finally, we showed the hedging strategy using the Brownian Gompertz stochastic force of mortality model and the results from Malliavin calculus.

Because of the medical improvement and desire to live long and healthy, life expectancy is getting longer and longer. Therefore, one of the key issues in insurance companies is how to model the improved mortality rate properly. There are still difficulties on designing a suitable mortality model. The best estimated model at a certain time cannot be used forever because mortality rates constantly change. For this reason, a natural way of hedging motality risk in life insurance using a stochastic mortality rate model is suggested in this paper. However, in reality, the difficulty of constructing a portfolio of term insurance and pure endowment should not be overlooked, because the term insurance market is relatively limited in South Korea.

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요약

미래시점에서의 사망률의 변화를 예측하는 것은 보험업계에서 중요하고 필요하다. 흥미로운 관측은 특정 연령의 사망률이 최근 개선되어서 사망률위험이 존재한다는 것이다. 생명표가 실제 생명보험 가입자의 사망률보다 낮게 예측한 사망모형을 기반으로 생성되었다면 그 회사는 생명보험 계약의 판매로부터 손실에 직면하게 될 것이다.
한 가지 헤지 방법으로서, 보험회사가 연금이나 생존보험과 같은 보험 상품의 판매를 촉진시키면 생명보험의 판매로부터 발생하는 손실을 상쇄할 수 있을 것이다. 이 논문에서는 생존보험과 정기 생명 보험으로 부터의 손실을 상쇄하는 헤지 비율을 이용한 저축성 생명보험 계약을 발전시킴으로 사망률 위험을 헤지할 수 있는 방법을 소개하겠다. 더 나아가, 말리아빈 적분으로부터의 결과를 이용한 확률 사망률 모델을 기반으로 헤지 방법을 보이도록 하겠다.

※ **국문 색인어:** 사망률위험, 사력, 헤지 비율, 확률 사망률 모델, 말리아빈 적분

Appendix

We assume that the dynamics of an asset $\{X(t):0\le t\le T\}$, which is an Rn-Markov process, are described by the stochastic differential equation,

$$dX(t)=\beta(X(t))dt+\sigma(X(t))dW(t), \tag{74}$$

where $\{W(t):0\le t\le T\}$ describes a Brownian motion with values in R^n.

We consider the price of a contingent claim defined by the following form

$$V(x)=E[\psi(X(t_1),\cdots,X(t_m))|X(0)=x] \tag{75}$$

where ψ is the payoff function on the times $0<t_1\le\cdots\le t_m=T$.

Next, we want to calculate the price of the path-dependent contingent claims and the sensitivity of $V(x)$ with respect to the initial condition x. We need to compute a Monte Carlo simulation of $V(x)$ and a Monte Carlo estimator $V(x+\epsilon)$ for a small, and estimate the sensitivity of $V(x)$ by the following value

$$\frac{\partial V}{\partial x}\approx\frac{V(x+\epsilon)-V(x)}{\epsilon}. \tag{76}$$

One way to calculate the sensitivity of $V(x)$ is to use Malliavin calculus. We briefly introduce a few resulting formulas needed in this paper. For more details on Malliavin calculus and its application to finance, refer to Bichteler et al.(1987), Malliavin(1997), Fournie et al.(2001, 1999), and Nualart(1995).

Using Malliavin calculus, it is well known that the differential of $V(x)$ can be expressed as

$$\frac{\partial V}{\partial x}=E[\pi\psi(X(t_1),\cdots X(t_m))|X(0)=x], \tag{77}$$

where π is a random variable to be determined.

There are many benefits of the above formula. We only need one Monte Carlo simulation to calculate the sensitivity of $V(x)$, which saves simulation and computing time. We also do not need the parallel shift of $x, x+\epsilon$, with an arbitrary small value of ϵ. We notice that the weight π does not depend on the payoff function ψ, which is also an important advantage of this formula.

Let us consider a probability space(Ω, F, P) and a set C of random variables on the Wiener-space Ω of the form

$$F = F(\omega) = f\left(\int_0^\infty h_1(t)dW(t), \cdots, \int_0^\infty h_n(t)dW(t)\right), \tag{78}$$

where ω is a path in the Wiener-space Ω, $f \in$ S(Rn), S(Rn) is the set of infinitely differentiable functions on R^n, and $h_1, \cdots, h_n \in L^2(\Omega \times R_+)$.

For $F \in C$, we define the Malliavin derivative DF of F by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^\infty h_1(t)dW(t), \cdots, \int_0^\infty h_n(t)dW(t)\right) h_i(t), \quad t \geq 0 \tag{79}$$

We define the norm of F by

$$\|F\|_{1,2} = \left(E[F^2]\right)^{1/2} + \left(E\left[\int_0^\infty (D_t F)^2 dt\right]\right)^{1/2} \tag{80}$$

We denote the Banach space which is the completion of C by $D^{1,2}$, with the norm $\| \bullet \|_{1,2}$. Let U be a stochastic process, $U(t) = U(\omega, t) \in L^2(\Omega \times R_+)$. For any $\phi \in D^{1,2}$ and fixed ω, both U and $D\phi$ are in $L^2(R_+) = H$. We use

$$\langle U, D\phi \rangle = \int_0^\infty U(t) D_t \phi dt, \tag{81}$$

for the standard inner product in $L^2(R_+)$. Note that this expression is stochastic,

$$\langle U, D\phi \rangle = \langle U, D\phi \rangle(\omega). \tag{82}$$

We calculate $E[\langle U, D\phi \rangle]$ by integrating over all $\omega \in \Omega$. We define $\delta(U)$,so called the Skorohod integral, as adjoint to D by

$$E[\langle U, D\phi \rangle] = E[\delta(U)\phi]. \qquad (83)$$

One of the interesting facts of Malliavin calculus is that the divergence operator coincides with the standard Itô integral. Let U be an adapted stochastic process in $L^2(\Omega \times R_+)$. Then we have

$$\delta(U) = \int_0^{\infty} U(t)\,dW(t). \qquad (84)$$

For an adapted random variable $F \in D^{1,2}$, using the chain rule for Malliavin derivatives,

$$D\psi(F) = \psi'(F)DF. \qquad (85)$$

Using integration by parts, we have

$$\begin{aligned} E[\psi'(F)] &= E\left[\left\langle D\psi(F), \frac{1}{\langle DF, DF\rangle} DF \right\rangle\right] \\ &= E\left[\psi(F)\delta\left(\frac{1}{\langle DF, DF\rangle} DF\right)\right] \\ &= E[\psi(F)\pi], \end{aligned} \qquad (86)$$

where $\pi = \delta\left(\frac{1}{\langle DF, DF\rangle} DF\right)$ is a random variable to be determined and ψ is a payoff function. Note that(86) is a way of obtaining the derivation of(77).